Filomat 30:1 (2016), 41–53
DOI 10.2298/FIL1601041N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Fundamental Properties of Fuzzy Linear Relations between Vector Spaces

Sorin Nădăban[a]

[a]*Aurel Vlaicu University of Arad, Department of Mathematics and Computer Science, Str. Elena Drăgoi 2, RO-310330 Arad, Romania*

Abstract. This paper aims at studying the fundamental properties of fuzzy linear relations between vector spaces. The sum of two fuzzy relations and the scalar multiplication are defined, in a natural way, and some properties of this operations are established. Fuzzy linear relations are investigated and among the results obtained, there should be underlined a characterization of fuzzy linear relations and the fact that the inverse of a fuzzy linear relation is also a fuzzy linear relation. Moreover, the paper shows that the composition of two fuzzy linear relations is a fuzzy linear relation as well. Finally, the article highlights that the family of all fuzzy linear relations is closed under addition and it is closed under scalar multiplication.

1. Introduction

Linear relations were introduced by R. Arens [1] in 1961. Since then, they have been a preoccupation for many mathematicians. It was only in 1998 that the theory of linear relations was systematized in a beautiful monograph by R. Cross [8]. The subject is not closed though. Thus, in 2003, A. Száz [15] investigated the possibility of extending of linear relation. The development of spectral theory for linear relations was the aims of recent papers: in 2002 A.G. Baskakov and K.I. Chernyshov [2], in 2007 A.G. Baskakov and A.S. Zagorskii [3], in 2012 D. Gheorghe and F.-H. Vasilescu [11]. It has to be mentioned that D. Gheorghe and F.-H. Vasilescu study in paper [10] linear maps defined between spaces of the form X/X_0, where X is a vector space and X_0 is a vector subspace of X. The motivation of this approach comes from the theory of linear relations.

On the other hand, the concept of fuzzy set introduced by L. Zadeh [19] in 1965, represented a natural frame for generalizing many of the concepts of mathematics. The introduction in 1977 by A.K. Katsaras and D.B. Liu [13] of the concept of fuzzy topological vector space resulted in what we can call today a new mathematical field ”Fuzzy Functional Analysis”. The present paper is based on results refereing to fuzzy linear subspaces obtained by A.K. Katsaras and D.B. Liu. In 1985, N.S. Papageorgiou [14] introduced the notion of fuzzy multifunction and started the study of linear fuzzy multifunction. The investigation of fuzzy multifunctions was continued, through a series of papers by E. Tsiporkova, B. De Baets, E. Kerre [17], [18] and I. Beg [4], [5], [6], [7]. A brief survey, concerning weakly linear systems of fuzzy relation inequalities and their applications in fuzzy automata, the study of simulation and in the social network analysis, was made in paper [12] by J. Ignjatović and M. Ćirić. There are also some recent papers in this

2010 *Mathematics Subject Classification*. Primary 47A06 ; Secondary 03E72, 47S40
Keywords. Fuzzy relation; fuzzy linear relation; fuzzy linear operator; fuzzy multivalued functions
Received: 12 April 2014; Accepted: 12 August 2015
Communicated by Hari M. Srivastava
Email address: snadaban@gmail.com (Sorin Nădăban)

field (see [9]). In [16], B. Šešelja, A. Tepavčevic and M. Udovičic are dealing with fuzzy posets and their fuzzy substructures.

The aim of this paper is to study the fundamental properties of fuzzy linear relations between vector spaces. Some results in the present paper may look similar, at the first sight, to I. Beg's results, but they are not. The differences between them are significant. Along this paper, we have identified fuzzy sets with their membership functions, while, I. Beg identified the fuzzy set with their support. Thus, in this paper, all equalities and inclusions are between fuzzy sets as compared to some of I. Beg's results where the obtained equalities (or inclusions) are only between the support sets of some fuzzy sets.

In this paper the sum of two fuzzy relations and the scalar multiplication are defined, in a natural way, and some properties of this operations are established. Fuzzy linear relations are investigated and among the results obtained, there should be underlined a characterization of fuzzy linear relations and the fact that the inverse of a fuzzy linear relation is also a fuzzy linear relation. Moreover, the paper shows that the composition of two fuzzy linear relations is a fuzzy linear relation as well. Finally, the article highlights that the family of all fuzzy linear relations is closed under addition and it is closed under scalar multiplication.

2. Fuzzy Linear Subspaces

Let X be a nonempty set. A *fuzzy set* in X (see [19]) is a function $\mu : X \to [0,1]$. We denote by $\mathcal{F}(X)$ the family of all fuzzy sets in X. The symbols $\vee$ and $\wedge$ are used for the supremum and infimum of a family of fuzzy sets. We write $\mu_1 \subseteq \mu_2$ if $\mu_1(x) \le \mu_2(x), (\forall)x \in X$. Let $f : X \to Y$. If $\mu \in \mathcal{F}(Y)$, then $f^{-1}(\mu) := \mu \circ f$. If $\rho \in \mathcal{F}(X)$, then $f(\rho) \in \mathcal{F}(Y)$ is defined by

$$f(\rho)(y) := \begin{cases} \vee\{\rho(x) \ : \ x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} .$$

If $\mu \in \mathcal{F}(X)$, the support of μ is $\operatorname{supp} \mu := \{x \in X \ : \ \mu(x) > 0\}$.

Let X be a vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). If $\mu_1, \mu_2, \cdots, \mu_n$ are fuzzy sets in X, then $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is a fuzzy set in X^n defined by (see [13])

$$\mu(x_1, x_2, \cdots, x_n) = \mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) .$$

Let $f : X^n \to X$, $f(x_1, x_2, \cdots, x_n) = \sum\limits_{k=1}^{n} x_k$. The fuzzy set $f(\mu)$ is called the *sum* of fuzzy sets $\mu_1, \mu_2, \cdots, \mu_n$ and it is denoted by $\mu_1 + \mu_2 + \cdots + \mu_n$ (see [13]). In fact

$$(\mu_1 + \mu_2 + \cdots + \mu_n)(x) = \bigvee \left\{ \mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) \ : \ x = \sum_{k=1}^{n} x_k \right\} .$$

If $\mu \in \mathcal{F}(X)$ and $\lambda \in \mathbb{K}$, then the fuzzy set $\lambda\mu$ is the image of μ under the map $g : X \to X$, $g(x) = \lambda x$. Thus (see [13])

$$(\lambda\mu)(x) = \begin{cases} \mu\left(\frac{x}{\lambda}\right) & \text{if } \ \lambda \neq 0 \\ 0 & \text{if } \ \lambda = 0, x \neq 0 \\ \vee\{\mu(y) \ : \ y \in X\} & \text{if } \ \lambda = 0, x = 0 \end{cases} .$$

Proposition 2.1. *Let $\mu_1, \mu_2 \in \mathcal{F}(X)$ and $\lambda \in \mathbb{K}$. Then $\lambda(\mu_1 + \mu_2) = \lambda\mu_1 + \lambda\mu_2$.*

Proof. It is obvious.

Definition 2.2. *[13] Let X be a vector space over $\mathbb{K}$. $\mu \in \mathcal{F}(X)$ is called fuzzy linear subspace of X if*

1. $\mu + \mu \subseteq \mu$;
2. $\lambda\mu \subseteq \mu, (\forall)\lambda \in \mathbb{K}$.

We de note by $FLS(X)$ the family of all fuzzy linear subspace of X.

Proposition 2.3. *[13] Let X be a vector space over $\mathbb{K}$ and $\mu \in \mathcal{F}(X)$. The following statements are equivalent:*

1. $\mu \in FLS(X)$;
2. $(\forall)\alpha, \beta \in \mathbb{K}$, *we have* $\alpha\mu + \beta\mu \subseteq \mu$;
3. $(\forall)x, y \in X$, $(\forall)\alpha, \beta \in \mathbb{K}$, *we have* $\mu(\alpha x + \beta y) \geq \mu(x) \wedge \mu(y)$.

Proposition 2.4. *[13] If $\mu, \rho \in FLS(X)$ and $\lambda \in \mathbb{K}$, then:*

1. $\mu + \rho \in FLS(X)$;
2. $\lambda\mu \in FLS(X)$;
3. $\mu(x) \leq \mu(0), (\forall)x \in X$.

3. Fuzzy Relations

In this section we consider X, Y be two nonempty sets.

A *fuzzy relation* T (or *fuzzy multifunction*, or *fuzzy multivalued function*) between X and Y is a fuzzy set in $X \times Y$, i.e. a mapping $T : X \times Y \to [0, 1]$. For $x \in X$, we denote by T_x the fuzzy set in Y defined by: $T_x : Y \to [0, 1], T_x(y) = T(x, y)$. Thus a fuzzy relation T can be seen as a mapping $X \ni x \mapsto T_x \in \mathcal{F}(Y)$ (see [14]). We denote by $FR(X, Y)$ the family of all fuzzy relations between X and Y. If $X = Y$, then we set $FR(X) = FR(X, Y)$.

The *domain* $D(T)$ of T is a fuzzy set in X defined by $D(T)(x) := \sup\limits_{y \in Y} T(x, y)$ (see [18]). We note that

$$\operatorname{supp} D(T) = \{x \in X \ : \ T_x \neq \emptyset\} = \{x \in X \ : \ (\exists)y \in Y \text{ such that } T(x, y) > 0\} .$$

If for all $x \in \operatorname{supp} D(T)$ there exists an unique $y \in Y$ such that $T(x, y) > 0$, then T is called *fuzzy function* (or *single-valued fuzzy function*). In this case, we denote this unique y by $T(x)$.

If $\mu \in \mathcal{F}(X)$, then $T(\mu) \in \mathcal{F}(Y)$ is defined by $T(\mu)(y) := \sup\limits_{x \in X}[T(x, y) \wedge \mu(x)]$ (see [4]). In particular, the *range* $R(T)$ of T is a fuzzy set in Y defined by $R(T)(y) := \sup\limits_{x \in X} T(x, y)$ (see [18]).

Let $T \in FR(X, Y), S \in FR(Y, Z)$. The *composition* $S \circ T \in FR(X, Z)$ (or simply ST) is defined by (see [19]): $(S \circ T)(x, z) := \sup\limits_{y \in Y}[T(x, y) \wedge S(y, z)]$.

Proposition 3.1. *Let $T \in FR(X, Y), S \in FR(Y, Z)$. Then $(S \circ T)_x = S(T_x), (\forall)x \in X$.*

Proof. It is obvious.

Proposition 3.2. *The operation " $\circ$ " is associative.*

Proof. It is obvious.

Definition 3.3. *Let $T \in FR(X, Y)$. If $E \subset X$, then the fuzzy relation $T|_E$ defined by*

$$T|_E : E \times Y \to [0, 1], T|_E(x, y) = T(x, y)$$

is called the restriction of T to E.

Moreover, the fuzzy relation T is called an extension to X of a fuzzy relation $S \in FR(E, Y)$ if $S = T|_E$.

The *inverse* (or *reverse relation*) T^{-1} of a fuzzy relation $T \in FR(X, Y)$ is a fuzzy set in $Y \times X$ defined by $T^{-1}(y, x) = T(x, y)$. It is obvious that $R(T) = D(T^{-1})$ and $R(T^{-1}) = D(T)$. We remark that, for $\mu \in \mathcal{F}(Y)$, we have $T^{-1}(\mu)(x) = \sup\limits_{y \in Y}[T^{-1}(y, x) \wedge \mu(x)] = \sup\limits_{y \in Y}[T(x, y) \wedge \mu(x)]$. This type of inverse is usually called *lower inverse* (see [4]).

$T \in FR(X, Y)$ is called *surjective* if $\operatorname{supp} R(T) = Y$. A fuzzy relation $T \in FR(X, Y)$ is called *injective* if, for $x_1 \neq x_2$, we have $T_{x_1} \wedge T_{x_2} = \emptyset$. This means that, for $x_1 \neq x_2$, we have $T(x_1, y) \wedge T(x_2, y) = 0, (\forall)y \in Y$.

A fuzzy relation $I \in FR(X)$ is called *identity relation* if $I(x, y) = 0$ for all $x, y \in X, x \neq y$. If we denote by E the support of $D(I)$, the identity relation will be denoted I_E. If $\mu \in \mathcal{F}(X)$, by identity relation on μ we understand $I_{\operatorname{supp} \mu}(x, x) = \mu(x)$.

Proposition 3.4. *Let $T \in FR(X, Y)$. Then*

1. *supp $D(T) = X$ if and only if $I_X \subseteq T^{-1}T$;*
2. *T is injective if and only if $T^{-1}T = I_{supp\ D(T)}$;*
3. *T is a fuzzy function if and only if $TT^{-1} = I_{supp\ R(T)}$;*
4. *T is injective if and only if T^{-1} is a fuzzy function.*

Proof. First we note that, for $x_1, x_2 \in X$, we have

$$T^{-1}T(x_1, x_2) = \sup_{y\in Y}[T(x_1, y) \wedge T^{-1}(y, x_2)] = \sup_{y\in Y}[T(x_1, y) \wedge T(x_2, y)].$$

1) For $x \in X$, we have

$$T^{-1}T(x, x) = \sup_{y\in Y}[T(x, y) \wedge T(x, y)] = \sup_{y\in Y} T(x, y) = D(T)(x).$$

Therefore

$$\text{supp } D(T) = X \Leftrightarrow D(T)(x) > 0, (\forall)x \in X \Leftrightarrow T^{-1}T(x, x) > 0, (\forall)x \in X \Leftrightarrow T^{-1}T \supseteq I_X.$$

2) $'' \Rightarrow ''$ For $x_1 \neq x_2$ we have $T(x_1, y) \wedge T(x_2, y) = 0, (\forall)y \in Y$. Therefore $T^{-1}T(x_1, x_2) = 0$. For $x \in X$ we have $T^{-1}T(x, x) = \sup_{y\in Y} T(x, y) = D(T)(x)$. Thus $T^{-1}T = I_{\text{supp } D(T)}$.

$'' \Leftarrow ''$ Let $x_1 \neq x_2$. As $T^{-1}T = I_{\text{supp } D(T)}$, we have that $T^{-1}T(x_1, x_2) = 0$. Thus, for all $y \in Y$, we have $T(x_1, y) \wedge T(x_2, y) = 0$. Hence $T_{x_1} \wedge T_{x_2} = \emptyset$.

3)

$$TT^{-1}(y_1, y_2) = \sup_{x\in X}[T^{-1}(y_1, x) \wedge T(x, y_2)] = \sup_{x\in X}[T(x, y_1) \wedge T(x, y_2)].$$

$'' \Rightarrow ''$ For $y_1 \neq y_2$, we have that $T(x, y_1) = 0$ or $T(x, y_2) = 0$, for all $x \in X$. Hence $TT^{-1}(y_1, y_2) = 0$. On the other hand $TT^{-1}(y, y) = \sup_{x\in X} T(x, y) = R(T)(y)$. Thus $TT^{-1} = I_{\text{supp } R(T)}$.

$'' \Leftarrow ''$ We suppose that T is not a fuzzy function. Then $(\exists)x \in \text{supp } D(T), (\exists)y_1, y_2 \in Y, y_1 \neq y_2$ such that $T(x, y_1) > 0, T(x, y_2) > 0$. Therefore $TT^{-1}(y_1, y_2) > 0$, contradiction.

4) T is injective $\Leftrightarrow T^{-1}T = I_{\text{supp } D(T)} \Leftrightarrow T^{-1}T = I_{\text{supp } R(T^{-1})} \Leftrightarrow T^{-1}$ is a fuzzy function.

4. Fuzzy Relations between Vector Spaces

In this section we consider X, Y be two vector spaces.

Definition 4.1. *Let $T, S \in FR(X, Y)$ and $\lambda \in \mathbb{K}$. We define the sum $T + S \in FR(X, Y)$ and scalar multiplication $\lambda T \in FR(X, Y)$ by*

$$(T + S)(x, y) = \sup_{y_1+y_2=y} [T(x, y_1) \wedge S(x, y_2)]$$

and

$$(\lambda T)(x, y) = T\left(x, \frac{y}{\lambda}\right) \text{ , if } \lambda \neq 0 \text{ ; } (0T)(x, y) = \begin{cases} D(T)x & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} .$$

For $\lambda \in \mathbb{K}$ we define the fuzzy function $\lambda_Y \in FR(Y)$ by

$$\lambda_Y(x, y) = \begin{cases} 1 & \text{if} \quad y = \lambda x \\ 0 & \text{if} \quad y \neq \lambda x \end{cases} .$$

In fact this is an ordinary function, precisely the function $Y \ni x \mapsto \lambda x \in Y$.

Proposition 4.2. *Let $T \in FR(X, Y)$ and $\lambda \in \mathbb{K}$. Then $\lambda T = \lambda_Y \circ T$.*

Proof. Case 1. $\lambda \neq 0$.

$$(\lambda_Y \circ T)(x,z) = \sup_{y\in Y}[T(x,y) \wedge \lambda_Y(y,z)] = T\left(x, \frac{z}{\lambda}\right) \wedge \lambda_Y\left(\frac{z}{\lambda}, z\right) = T\left(x, \frac{z}{\lambda}\right) = (\lambda T)(x,z)\ .$$

Case 2. $\lambda = 0$.

$$(\lambda_Y \circ T)(x,z) = \sup_{y\in Y}[T(x,y) \wedge \lambda_Y(y,z)] = \begin{cases} \sup\limits_{y\in Y} T(x,y) & \text{if} \quad z = 0 \\ 0 & \text{if} \quad z \neq 0 \end{cases} =$$

$$= \begin{cases} D(T)x & \text{if} \quad z = 0 \\ 0 & \text{if} \quad z \neq 0 \end{cases} = (\lambda T)(x,z)\ .$$

Proposition 4.3. *Let $T, S \in FR(X,Y)$ and $\lambda \in \mathbb{K}$. Then*

1. $D(T+S) = D(T) \wedge D(S)$;
2. $D(\lambda T) = D(T)$.

Proof. 1)

$$D(T+S)(x) = \sup_{y\in Y}(T+S)(x,y) = \sup_{y\in Y}\ \sup_{y_1+y_2=y}\ [T(x,y_1) \wedge S(x,y_2)] = \sup_{y_1,y_2\in Y}[T(x,y_1) \wedge S(x,y_2)] =$$

$$= \sup_{y_1\in Y} T(x,y_1) \wedge \sup_{y_2\in Y} S(x,y_2) = D(T)x \wedge D(S)x = [D(T) \wedge D(S)](x)\ .$$

2) Case 1. $\lambda \neq 0$.

$$D(\lambda T)(x) = \sup_{y\in Y}(\lambda T)(x,y) = \sup_{y\in Y} T\left(x, \frac{y}{\lambda}\right) = \sup_{y'\in Y} T(x,y') = D(T)x\ .$$

Case 2. $\lambda = 0$.

$$D(0T)x = \sup_{y\in Y}(0T)(x,y) = (0T)(x,0) = D(T)x\ .$$

Proposition 4.4. *Let $T \in FR(X,Y)$ and $\lambda \in \mathbb{K}$. Let* $\sup T := \sup\{T(x,y) \ :\ x \in X, y \in Y\}$. *Then* $\sup \lambda T = \sup T$.

Proof. Case 1. $\lambda \neq 0$.

$$\sup \lambda T := \sup\{(\lambda T)(x,y) \ :\ x \in X, y \in Y\} = \sup\{T\left(x, \frac{y}{\lambda}\right) \ :\ x \in X, y \in Y\} =$$

$$= \sup\{T(x,y') \ :\ x \in X, y' \in Y\} = \sup T\ .$$

Case 2. $\lambda = 0$.

$$\sup \lambda T := \sup\{(0T)(x,y) \ :\ x \in X, y \in Y\} = \sup_{x\in X}(0T)(x,0) =$$

$$= \sup_{x\in X} D(T)x = \sup_{x\in X}\sup_{y\in Y} T(x,y) = \sup T\ .$$

Theorem 4.5. *Let $T, S, R \in FR(X,Y)$ and $\alpha, \beta \in \mathbb{K}$. Then*

1. $(T+S)+R = T+(S+R)$;
2. $T+S = S+T$;
3. $T+0 = T$, *where* $0 \in FR(X,Y)$ *is defined by* $0(x,y) = \begin{cases} 1 & \textit{if} \quad y = 0 \\ 0 & \textit{if} \quad y \neq 0 \end{cases}$;
4. $\alpha(T+S) = \alpha T + \alpha S$;
5. $\alpha(\beta T) = (\alpha\beta)T$;
6. $1T = T$.

Proof. 1) Let $(x, y) \in X \times Y$. Then

$$[(T+S)+R](x,y) = \sup_{z+t=y}[(T+S)(x,z) \wedge R(x,t)] =$$

$$= \sup_{z+t=y}[[\sup_{h+u=z} T(x,h) \wedge S(x,u)] \wedge R(x,t)] = \sup_{h+u+t=y} [T(x,h) \wedge S(x,u) \wedge R(x,t)] .$$

$$[T+(S+R)](x,y) = \sup_{h+z=y}[T(x,h) \wedge (S+R)(x,z)] =$$

$$= \sup_{h+z=y}[T(x,h) \wedge [\sup_{u+t=z} S(x,u) \wedge R(x,t)]] = \sup_{h+u+t=y} [T(x,h) \wedge S(x,u) \wedge R(x,t)] .$$

2) and 3) are obvious.
4) Case 1. $\alpha \neq 0$.

$$[\alpha(T+S)](x,y) = (T+S)\left(x, \frac{y}{\alpha}\right) = \sup_{\frac{y_1}{\alpha}+\frac{y_2}{\alpha}=\frac{y}{\alpha}} \left[T\left(x, \frac{y_1}{\alpha}\right) \wedge S\left(x, \frac{y_2}{\alpha}\right)\right] =$$

$$= \sup_{y_1+y_2=y} [(\alpha T)(x,y_1) \wedge (\alpha S)(x,y_2)] = (\alpha T + \alpha S)(x,y) .$$

Case 2. $\alpha = 0$.

$$(0T+0S)(x,y) = \sup_{y_1+y_2=y} [(0T)(x,y_1) \wedge (0S)(x,y_2)] .$$

For $y \neq 0$, we have $y_1 \neq 0$ or $y_2 \neq 0$. Thus $(0T)(x,y_1) \wedge (0S)(x,y_2) = 0$. Hence $(0T+0S)(x,y) = 0$. For $y = 0$, we have

$$\sup_{y_1+y_2=y} [(0T)(x,y_1) \wedge (0S)(x,y_2)] = (0T)(x,0) \wedge (0S)(x,0) =$$

$$= D(T)x \wedge D(S)x = (D(T) \wedge D(S))(x) = D(T+S)x .$$

Therefore

$$(0T+0S)(x,y) = \begin{cases} D(T+S)x & \text{if} \quad y=0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} = [0(T+S)](x,y) .$$

5) Case 1. $\alpha \neq 0, \beta \neq 0$.

$$[\alpha(\beta T)](x,y) = (\beta T)\left(x, \frac{y}{\alpha}\right) = T\left(x, \frac{y}{\alpha\beta}\right) = [(\alpha\beta)T](x,y) .$$

Case 2. $\alpha = 0$.

$$[\alpha(\beta T)](x,y) = \begin{cases} D(\beta T)x & \text{if} \quad y=0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} =$$

$$= \begin{cases} D(T)x & \text{if} \quad y=0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} = (0T)(x,y) = [(\alpha\beta)T](x,y) .$$

Case 3. $\alpha \neq 0, \beta = 0$.

$$[\alpha(\beta T)](x,y) = (\beta T)\left(x, \frac{y}{\alpha}\right) = \begin{cases} D(T)x & \text{if} \quad y=0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} = (0T)(x,y) = [(\alpha\beta)T](x,y) .$$

6) It is obvious.

5. Fuzzy Linear Relations

Definition 5.1. *Let X, Y be two vector spaces over $\mathbb{K}$. A fuzzy linear relation (or fuzzy linear multivalued operator) between X and Y is a fuzzy linear subspace in $X \times Y$.*

We denote by $FLR(X, Y)$ the family of all fuzzy linear relations between X and Y. If $X = Y$, then we set $FLR(X) = FLR(X, X)$.

If $T \in FLR(X, Y)$ is a single-valued fuzzy function, then T will be called fuzzy linear operator. We denote by $FLO(X, Y)$ the family of all fuzzy linear operators between X and Y. In the case $X = Y$ the family $FLO(X, X)$ will be denoted $FLO(X)$.

Remark 5.2. *Using Proposition 2.3, the linearity of $T \in FLR(X, Y)$ can be written as*

$$T(\alpha(x_1, y_1) + \beta(x_2, y_2)) \geq T(x_1, y_1) \wedge T(x_2, y_2)\ , (\forall)\alpha, \beta \in \mathbb{K}, (\forall)(x_1, y_1), (x_2, y_2) \in X \times Y\ .$$

We also note that, for $\beta = 0$, we obtain $T(\alpha(x_1, y_1)) \geq T(x_1, y_1)$.

Remark 5.3. *Proposition 2.4 implies that $T(x, y) \leq T(0, 0), (\forall)x \in X, y \in Y$.*

Theorem 5.4. *Let $T \in FR(X, Y)$. Then $T \in FLR(X, Y)$ if and only if*

1. $T_{x_1} + T_{x_2} \subseteq T_{x_1+x_2}, (\forall)x_1, x_2 \in X$;
2. $\alpha T_x \subseteq T_{\alpha x}, (\forall)x \in X, (\forall)\alpha \in \mathbb{K}$.

Proof. $'' \Rightarrow ''$ 1) Let $y \in Y$. Then

$$T_{x_1+x_2}(y) = T(x_1 + x_2, y) = T(x_1 + x_2, y_1 + y - y_1) \geq T(x_1, y_1) \wedge T(x_2, y - y_1)\ .$$

Therefore

$$T_{x_1+x_2}(y) \geq \sup_{y_1 \in Y}[T(x_1, y_1) \wedge T(x_2, y - y_1)] = (T_{x_1} + T_{x_2})(y)\ .$$

Hence $T_{x_1+x_2} \supseteq T_{x_1} + T_{x_2}$.
2) First we note that

$$(\alpha T_x)(y) = \begin{cases} T_x\left(\frac{y}{\alpha}\right) & \text{if} \quad \alpha \neq 0 \\ 0 & \text{if} \quad \alpha = 0, y \neq 0 \\ \vee\{T_x(z)\ :\ z \in Y\} & \text{if} \quad \alpha = 0, y = 0 \end{cases} .$$

Case 1. $\alpha \neq 0$.

$$(T_{\alpha x})(y) = T(\alpha x, y) = T\left(\alpha\left(x, \frac{y}{\alpha}\right)\right) \geq T\left(x, \frac{y}{\alpha}\right) = T_x\left(\frac{y}{\alpha}\right) = (\alpha T_x)(y)\ .$$

Case 2. $\alpha = 0$. If $y \neq 0$, as $(\alpha T_x)(y) = 0$, we have that $(\alpha T_x)(y) \leq (T_{\alpha x})(y)$. For $y = 0$ we have

$$(\alpha T_x)(y) = \vee\{T_x(z)\ :\ z \in Y\} \leq T(0, 0) = T(\alpha x, y) = T_{\alpha x}(y)\ .$$

Hence $(\alpha T_x)(y) \leq T_{\alpha x}(y), (\forall)y \in Y$. Thus $\alpha T_x \subseteq T_{\alpha x}$.
$'' \Leftarrow ''$ Let $(x_1 + x_2, y) \in X \times Y$. Then

$$T(x_1 + x_2, y) = T_{x_1+x_2}(y) \geq (T_{x_1} + T_{x_2})(y) = \sup_{y_1+y_2=y}[T_{x_1}(y_1) \wedge T_{x_2}(y_2)]\ .$$

Therefore

$$T(x_1 + x_2, y_1 + y_2) \geq T_{x_1}(y_1) \wedge T_{x_2}(y_2)\ .$$

On the other hand, for $\alpha \neq 0$, we have $(\alpha T_x)(\alpha y) = T_x(y) = T(x, y)$. As $\alpha T_x \subseteq T_{\alpha x}$, we obtain that $(\alpha T_x)(\alpha y) \leq T_{\alpha x}(\alpha y)$, namely $T(x, y) \leq T(\alpha(x, y))$. If $\alpha = 0$, we have that

$$(\alpha T_x)(\alpha y) = \vee\{T_x(z)\ :\ z \in Y\} = \vee\{T(x, z)\ :\ z \in Y\}$$

and

$$T_{\alpha x}(\alpha y) = T(\alpha x, \alpha y) = T(0,0) \ .$$

Therefore $\alpha T_x \subseteq T_{\alpha x}$ implies that $\vee\{T(x,z) \ : \ z \in Y\} \leq T(0,0)$, namely $T(x,z) \leq T(0,0)$, for all $x \in X, z \in Y$. Now, we will prove that

$$T(\alpha(x_1,y_1)+\beta(x_2,y_2)) \geq T(x_1,y_1) \wedge T(x_2,y_2) \ , (\forall)\alpha,\beta \in \mathbb{K}, (\forall)(x_1,y_1),(x_2,y_2) \in X \times Y \ .$$

Case 1. $\alpha \neq 0, \beta \neq 0$.

$$T(\alpha(x_1,y_1)+\beta(x_2,y_2)) \geq T(\alpha x_1, \alpha y_1) \wedge T(\beta x_2, \beta y_2) \geq T(x_1,y_1) \wedge T(x_2,y_2) \ .$$

Case 2. $\alpha = 0, \beta \neq 0$.

$$T(\alpha(x_1,y_1)+\beta(x_2,y_2)) = T(\beta(x_2,y_2)) \geq T(x_2,y_2) \geq T(x_1,y_1) \wedge T(x_2,y_2) \ .$$

Case 3. $\alpha \neq 0, \beta = 0$. Similarly to the previous case.
Case 4. $\alpha = 0, \beta = 0$.

$$T(\alpha(x_1,y_1)+\beta(x_2,y_2)) = T(0,0) \geq T(x_1,y_1) \wedge T(x_2,y_2) \ .$$

Corollary 5.5. *If $T \in FLR(X,Y)$, then T_0 is a fuzzy linear subspace of Y.*

Proof. By previous theorem we have that $T_0 + T_0 \subseteq T_0$, $\alpha T_0 \subseteq T_0$. This means that T_0 is a fuzzy linear subspace of Y.

Theorem 5.6. *Let $T \in FR(X,Y)$. Then $T \in FLR(X,Y)$ if and only if $T^{-1} \in FLR(Y,X)$.*

Proof. Let $\alpha, \beta \in \mathbb{K}$, $(y_1,x_1),(y_2,x_2) \in Y \times X$. Then

$$T^{-1}(\alpha(y_1,x_1)+\beta(y_2,x_2)) = T^{-1}(\alpha y_1+\beta y_2, \alpha x_1+\beta x_2) = T(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2) =$$

$$= T(\alpha(x_1,y_1)+\beta(x_2,y_2) \geq T(x_1,y_1) \wedge T(x_2,y_2) = T^{-1}(y_1,x_1) \wedge T^{-1}(y_2,x_2) \ .$$

Similarly, we can prove that if $T^{-1} \in FLR(Y,X)$, then $T \in FLR(X,Y)$.

Corollary 5.7. *If $T \in FLR(X,Y)$, then T_0^{-1} is a fuzzy linear subspace of X.*

Definition 5.8. *If $T \in FLR(X,Y)$, then T_0^{-1} is called the kernel of T and it is denoted $Ker(T)$.*

Proposition 5.9. *Let $T \in FLO(X,Y)$ nonempty. Then*

1. $T(x_1)+T(x_2) = T(x_1+x_2), (\forall)x_1,x_2 \in D(T)$;
2. $\alpha T(x) = T(\alpha x), (\forall)x \in D(T), (\forall)\alpha \in \mathbb{K}$.

Proof. 1)

$$(T_{x_1}+T_{x_2})(T(x_1)+T(x_2)) = \sup\{T_{x_1}(y_1) \wedge T_{x_2}(y_2) \ : y_1+y_2 = T(x_1)+T(x_2)\} =$$

$$= T_{x_1}(T(x_1)) \wedge T_{x_2}(T(x_2)) > 0 \ .$$

Therefore $T_{x_1+x_2}(T(x_1)+T(x_2)) > 0$, namely $T(x_1)+T(x_2) = T(x_1+x_2)$.
2) For $\alpha \neq 0$, we have $(\alpha T_x)(\alpha T(x)) = T_x(T(x)) > 0$. Thus $T_{\alpha x}(\alpha T(x)) > 0$. Hence $\alpha T(x) = T(\alpha x)$. If $\alpha = 0$, we must prove that $T(0) = 0$, namely $T(0,0) > 0$. But

$$T(0,0) \geq T(x,y), (\forall)x \in X, (\forall)y \in Y \ .$$

Therefore, if $T(0,0) = 0$, we obtain that $T(x,y), (\forall)x \in X, (\forall)y \in Y$, i.e. T is the empty set.

Theorem 5.10. *If $T \in FLR(X,Y)$, $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$ and $\lambda \in \mathbb{K}$, then*

1. $T(\mu_1)+T(\mu_2) \subseteq T(\mu_1+\mu_2)$;

2. $\mu_1 \subseteq \mu_2 \Rightarrow T(\mu_1) \subseteq T(\mu_2)$;
3. $\lambda T(\mu) \subseteq T(\lambda\mu)$ *and* $\lambda T(\mu) = T(\lambda\mu)$, *for* $\lambda \neq 0$.

Proof. 1) Let $y \in Y$ fixed. We will prove that

$$[T(\mu_1) + T(\mu_2)](y) \leq T(\mu_1 + \mu_2)(y) .$$

If $[T(\mu_1) + T(\mu_2)](y) = 0$, then the previous inequality is obvious. We suppose that

$$A = [T(\mu_1) + T(\mu_2)](y) > 0 .$$

As

$$A = [T(\mu_1) + T(\mu_2)](y) = \sup_{y_1+y_2=y} [T(\mu_1)(y_1) + T(\mu_2)(y_2)] ,$$

for $\varepsilon \in (0, A)$ arbitrary, $(\exists) y_1, y_2 \in Y \ : \ y_1 + y_2 = y$, such that

$$T(\mu_1)(y_1) + T(\mu_2)(y_2) > A - \varepsilon .$$

But $T(\mu_1)(y_1) = \sup_{x \in X}[T(x, y_1) \wedge \mu_1(x)] > A - \varepsilon$ implies that there exists $x_1 \in X$ such that

$$T(x_1, y_1) > A - \varepsilon, \mu_1(x_1) > A - \varepsilon .$$

On the other hand $T(\mu_2)(y_2) = \sup_{x \in X}[T(x, y_2) \wedge \mu_2(x)] > A - \varepsilon$ implies that there exists $x_2 \in X$ such that $T(x_2, y_2) > A - \varepsilon, \mu_2(x_2) > A - \varepsilon$. Then

$$T(x_1 + x_2, y_1 + y_2) \geq T(x_1, y_1) \wedge T(x_2, y_2) > A - \varepsilon$$

and

$$(\mu_1 + \mu_2)(x_1 + x_2) = \sup_{x_1'+x_2'=x_1+x_2} [\mu_1(x_1') \wedge \mu_2(x_2')] \geq \mu_1(x_1) \wedge \mu_2(x_2) > A - \varepsilon .$$

Thus

$$B = T(\mu_1 + \mu_2)(y) = \sup_{x \in X}[T(x, y) \wedge (\mu_1 + \mu_2)(x)] \geq T(x_1 + x_2, y) \wedge (\mu_1 + \mu_2)(x_1 + x_2) > A - \varepsilon .$$

Hence $B > A - \varepsilon$. As ε is arbitrary, we obtain that $B \geq A$, i.e. the desired inequality.
2) Let $y \in Y$. Then

$$T(\mu_1)(y) = \sup_{x \in X}[T(x, y) \wedge \mu_1(x)] \leq \sup_{x \in X}[T(x, y) \wedge \mu_2(x)] = T(\mu_2)(y) .$$

3) Case 1. $\lambda \neq 0$. First, we note that

$$T\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) = T\left(\frac{1}{\lambda}(x, y)\right) \geq T(x, y) .$$

On the other hand

$$T(x, y) = T\left(\lambda\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right)\right) \geq T\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) .$$

Thus

$$T\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) = T(x, y) .$$

Therefore

$$[\lambda T(\mu)](y) = T(\mu)\left(\frac{y}{\lambda}\right) = \sup_{x \in X}\left[T\left(x, \frac{y}{\lambda}\right) \wedge \mu(x)\right] = \sup_{x \in X}\left[T\left(x, \frac{y}{\lambda}\right) \wedge (\lambda\mu)(\lambda x)\right] =$$

$$= \sup_{x\in X}\left[T\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)\wedge(\lambda\mu)(x)\right] = \sup_{x\in X}[T(x,y)\wedge(\lambda\mu)(x)] = [T(\lambda\mu)](y)\ .$$

Case 2. $\lambda = 0$. If $y \neq 0$, then $[\lambda T(\mu)](y) = 0$. Thus $[\lambda T(\mu)](y) \leq [T(\lambda\mu)](y)$. If $y = 0$, we have

$$[\lambda T(\mu)](y) = \sup_{z\in Y} T(\mu)(z) = \sup_{z\in Y}\sup_{x\in X}[T(x,z)\wedge\mu(x)]\ .$$

As $T(x,z) \leq T(0,0), (\forall)x \in X, (\forall)z \in Y$, we obtain that

$$[\lambda T(\mu)](y) \leq T(0,0)\wedge\sup_{x\in X}\mu(x)\ .$$

But

$$[T(\lambda\mu)](y) = [T(\lambda\mu)](0) = \sup_{x\in X}[T(x,0)\wedge(\lambda\mu)(x)]\ .$$

As $\lambda = 0$, for $x \neq 0$, we have that $(\lambda\mu)(x) = 0$. Therefore

$$[T(\lambda\mu)](y) = T(0,0)\wedge(\lambda\mu)(0) = T(0,0)\wedge\sup_{x\in X}\mu(x)\ .$$

Hence $[\lambda T(\mu)](y) \leq [T(\lambda\mu)](y), (\forall)y \in Y$.

Theorem 5.11. *If $T \in FLR(X,Y), S \in FLR(Y,Z)$, then $S\circ T \in FLR(X,Z)$.*

Proof. Let $x_1, x_2 \in X$. Then

$$(S\circ T)_{x_1} + (S\circ T)_{x_2} = S(T_{x_1}) + S(T_{x_2}) \subseteq S(T_{x_1}+T_{x_2}) \subseteq S(T_{x_1+x_2}) = (S\circ T)_{x_1+x_2}\ .$$

Let $x \in X$ and $\lambda \in \mathbb{K}$. Then

$$\lambda(S\circ T)_x = \lambda S(T_x) \subseteq S(\lambda T_x) \subseteq S(T_{\lambda x}) = (S\circ T)_{\lambda x}\ .$$

Proposition 5.12. *Let $T \in FLR(X,Y)$ nonempty. Then T is single valued fuzzy function if and only if supp $T_0 = \{0\}$.*

First, we note that $0 \in$ supp T_0, i.e. $T(0,0) > 0$, contrary T is empty set.
$'' \Rightarrow ''$ If there exists $x \neq 0, x \in$ supp T_0, we obtain that $T_0(x) > 0$, namely $T(0,x) > 0$, contradiction with the fact that T is single-valued.
$'' \Leftarrow ''$ Let $x \in D(T)$ fixed. Then there exists $y \in Y\ :\ T(x,y) > 0$. We suppose that T is not a single-valued fuzzy function. Then there exists $y' \in Y, y' \neq y\ :\ T(x,y') > 0$. As

$$T(x,y)\wedge T(x,y') \leq T((x,y)-(x,y')) = T(0,y-y') = 0\ ,$$

we have that $T(x,y)\wedge T(x,y') = 0$, contradiction.

Proposition 5.13. *Let $T \in FLR(X,Y)$. Then*

1. $T(T_0^{-1}) = T_0$;
2. $T^{-1}(T_0) = T_0^{-1}$.

Proof. 1)

$$T(T_0^{-1})(y) = \sup_{x\in X}[T(x,y)\wedge T_0^{-1}(x)] = \sup_{x\in X}[T(x,y)\wedge T(x,0)]\ .$$

First we note that

$$T(T_0^{-1})(y) = \sup_{x\in X}[T(x,y)\wedge T(x,0)] \geq T(0,y)\wedge T(0,0) = T(0,y)\ .$$

On the other hand, as $T(x,y)\wedge T(x,0) \leq T((x,y)-(x,0)) = T(0,y)$, we have

$$T(T_0^{-1})(y) = \sup_{x\in X}[T(x,y)\wedge T(x,0)] \leq \sup_{x\in X} T(0,y) = T(0,y)\ .$$

Therefore

$$T(T_0^{-1})(y) = T(0,y) = T_0(y), (\forall)y \in Y\ .$$

2) The proof is similar.

Proposition 5.14. *Let $T \in FLR(X, Y)$ nonempty. Then T is injective if and only if supp $Ker(T) = \{0\}$.*

Proof. As $T(x, y) \leq T(0, 0), (\forall)x \in X, (\forall)y \in Y$, we have that $T(0, 0) > 0$, contrary T is empty set. This implies that $0 \in$ supp $Ker(T)$.
$'' \Rightarrow ''$ We suppose that there exists $x \neq 0\ : x \in$ supp $Ker(T)$, i.e. $T(x, 0) > 0$. As T is injective and $x \neq 0$, we have that $T_x \wedge T_0 = \emptyset$, namely $T(x, y) \wedge T(0, y) = 0, (\forall)y \in Y$. Particulary, for $y = 0$, we have $T(x, 0) \wedge T(0, 0) = 0$, contradiction.
$'' \Leftarrow ''$ Let $x_1 \neq x_2$. Then $x_1 - x_2 \neq 0$. Therefore $x_1 - x_2 \notin$ supp $Ker(T)$, namely $T(x_1 - x_2, 0) = 0$. We will prove that $T_{x_1} \wedge T_{x_2} = \emptyset$. Indeed, for $y \in Y$, we have

$$T_{x_1}(y) \wedge T_{x_2}(y) = T(x_1, y) \wedge T(x_2, y) \leq T((x_1, y) - (x_2, y)) = T(x_1 - x_2, 0) = 0\ .$$

Thus $T_{x_1}(y) \wedge T_{x_2}(y) = 0, (\forall)y \in Y$, i.e. $T_{x_1} \wedge T_{x_2} = \emptyset$. Therefore T is injective.

Proposition 5.15. *If $T \in FLR(X, Y)$, then $T_x = T_x + T_0, (\forall)x \in X$.*

Proof. Let $y \in Y$. Then $(T_x + T_0)(y) = \sup\limits_{y_1+y_2=y} [T_x(y_1) \wedge T_0(y_2)] \geq T_x(y) \wedge T_0(0) = T(x, y)$. On the other hand

$$T_x(y_1) \wedge T_0(y_2) = T(x, y_1) \wedge T(0, y_2) \leq T((x, y_1) + (0, y_2)) = T(x, y)\ .$$

Hence

$$(T_x + T_0)(y) = \sup_{y_1+y_2=y} [T_x(y_1) \wedge T_0(y_2)] \leq T(x, y)\ .$$

Therefore $(T_x + T_0)(y) = T(x, y) = T_x(y)$, i.e. $T_x + T_0 = T_0$.

Theorem 5.16. *$FLR(X, Y)$ is closed under addition.*

Proof. Let $T, S \in FLR(X, Y)$. We will prove that $T + S \in FLR(X, Y)$. First, we note that

$$(T + S)_x(y) = \sup_{y_1+y_2=y} [T_x(y_1) \wedge S_x(y_2)] = (T_x + S_x)(y)\ .$$

Hence $(T + S)_x = T_x + S_x$. Therefore

$$(T + S)_{x_1} + (T + S)_{x_2} = T_{x_1} + S_{x_1} + T_{x_2} + S_{x_2} \subseteq T_{x_1+x_2} + S_{x_1+x_2} = (T + S)_{x_1+x_2}\ .$$

Using Proposition 2.1, we obtain

$$\alpha(T + S)_x = \alpha(T_x + S_x) = \alpha T_x + \alpha S_x \subseteq T_{\alpha x} + S_{\alpha x} = (T + S)_{\alpha x}\ .$$

Theorem 5.17. *$FLR(X, Y)$ is closed under scalar multiplication.*

Proof. Let $T \in FLR(X, Y)$ and $\lambda \in \mathbb{K}$. We will prove that $\lambda T \in FLR(X, Y)$.
Case 1. $\lambda \neq 0$. As $(\lambda T)(x, y) = T\left(x, \frac{y}{\lambda}\right)$, we obtain that $(\lambda T)_x(y) = T_x\left(\frac{y}{\lambda}\right)$. Thus

$$[(\lambda T)_{x_1} + (\lambda T)_{x_2}](y) = \sup_{y_1+y_2=y} [(\lambda T)_{x_1}(y_1) \wedge (\lambda T)_{x_2}(y_2)] = \sup_{y_1+y_2=y} \left[T_{x_1}\left(\frac{y_1}{\lambda}\right) \wedge T_{x_2}\left(\frac{y_2}{\lambda}\right)\right] =$$

$$= \sup_{\frac{y_1}{\lambda}+\frac{y_2}{\lambda}=\frac{y}{\lambda}} \left[T_{x_1}\left(\frac{y_1}{\lambda}\right) \wedge T_{x_2}\left(\frac{y_2}{\lambda}\right)\right] = (T_{x_1} + T_{x_2})\left(\frac{y}{\lambda}\right) \leq T_{x_1+x_2}\left(\frac{y}{\lambda}\right) = (\lambda T)_{x_1+x_2}(y)\ .$$

Hence $(\lambda T)_{x_1} + (\lambda T)_{x_2} \subseteq (\lambda T)_{x_1+x_2}$. Now we prove that $\alpha(\lambda T)_x \subseteq (\lambda T)_{\alpha x}$.
Case 1a. $\alpha \neq 0$. Let $y \in Y$. Then

$$[\alpha(\lambda T)_x](y) = (\lambda T)_x\left(\frac{y}{\alpha}\right) = (\lambda T)\left(x, \frac{y}{\alpha}\right) = T\left(x, \frac{y}{\alpha\lambda}\right) =$$

$$= T_x\left(\frac{y}{\alpha\lambda}\right) = [\alpha T_x]\left(\frac{y}{\lambda}\right) \leq T_{\alpha x}\left(\frac{y}{\lambda}\right) = (\lambda T)_{\alpha x}(y)\ .$$

Case 1b. $\alpha = 0$. For $y \neq 0$, we have that $[0(\lambda T)_x](y) = 0$. Thus $[0(\lambda T)_x](y) \leq (\lambda T)_{\alpha x}(y)$. If $y = 0$, we have

$$[0(\lambda T)_x](0) = \sup_{z\in Y}(\lambda T)_x(z) = \sup_{z\in Y}(\lambda T)(x,z) =$$

$$= D(\lambda T)(x) = D(T)(x) = \sup_{z\in Y} T(x,z) \leq T(0,0) = (\lambda T)(0,0) = (\lambda T)_{0x}(0)\ .$$

Hence $[0(\lambda T)_x](y) \leq (\lambda T)_{0x}(y), (\forall) y \in Y$. Thus $0(\lambda T)_x \subseteq (\lambda T)_{0x}$.

Case 2. $\lambda = 0$. Then

$$(0T)(x,y) = \begin{cases} D(T)x & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}\ .$$

Therefore

$$[(0T)_{x_1} + (0T)_{x_2}](y) = \sup_{y_1+y_2=y}[(0T)_{x_1}(y_1) \wedge (0T)_{x_2}(y_2)] = \sup_{y_1+y_2=y}[(0T)(x_1,y_1) \wedge (0T)(x_2,y_2)] =$$

$$= \begin{cases} (0T)(x_1,0) \wedge (0T)(x_2,0) & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} = \begin{cases} D(T)(x_1) \wedge D(T)(x_2) & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}\ .$$

On the other hand

$$(0T)_{x_1+x_2}(y) = (0T)(x_1+x_2,y) = \begin{cases} D(T)(x_1+x_2) & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}\ .$$

We remark that $D(T)(x_1) \wedge D(T)(x_2) \leq D(T)(x_1+x_2)$ and thus $(0T)_{x_1} + (0T)_{x_2} \subseteq (0T)_{x_1+x_2}$. Indeed,

$$D(T)(x_1) \wedge D(T)(x_2) = \sup_{y_1\in Y} T(x_1,y_1) \wedge \sup_{y_2\in Y} T(x_2,y_2) =$$

$$= \sup_{y_1,y_2\in Y} T(x_1,y_1) \wedge T(x_2,y_2) \leq \sup_{y_1,y_2\in Y} T(x_1+x_2,y_1+y_2) = D(T)(x_1+x_2)\ .$$

It rest to show that $\alpha(0T)_x \subseteq (0T)_{\alpha x}$.

Case 2a. $\alpha \neq 0$. Then

$$[\alpha(0T)_x](y) = (0T)_x\left(\frac{y}{\alpha}\right) = (0T)\left(x,\frac{y}{\alpha}\right) = \begin{cases} D(T)x & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases} = (0T)(x,y)\ .$$

On the other hand

$$(0T)_{\alpha x}(y) = (0T)(\alpha x,y) = \begin{cases} D(T)(\alpha x) & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}\ .$$

In order to establish the inclusion $\alpha(0T)_x \subseteq (0T)_{\alpha x}$, we must show that $D(T)x \leq D(T)(\alpha x)$. But

$$D(T)(\alpha x) = \sup_{y\in Y} T(\alpha x,y) = \sup_{y\in Y} T\left(\alpha\left(x,\frac{y}{\alpha}\right)\right) \geq \sup_{y\in Y} T\left(x,\frac{y}{\alpha}\right) = \sup_{y'\in Y} T(x,y') = D(T)x\ .$$

Case 2b. $\alpha = 0$. For $y \neq 0$, we have $[0(0T)_x](y) = 0$. Thus $[0(0T)_x](y) \leq (0T)_{0x}(y)$. If $y = 0$, then

$$[0(0T)_x](0) = \sup_{z\in Z}(0T)_x(z) = \sup_{z\in Z}(0T)(x,z) = (0T)(x,0) = D(T)(x)$$

and

$$(0T)_{0x}(0) = (0T)(0,0) = D(T)(0) = \sup_{y\in Y} T(0,y) = T(0,0)\ .$$

As $D(T)(x) = \sup_{y\in Y} T(x,y) \leq T(0,0)$, we obtain that $[0(0T)_x](0) \leq (0T)_{0x}(0)$. Thus

$$[0(0T)_x](y) \leq (0T)_{0x}(y), (\forall) y \in Y\ .$$

Hence $0(0T)_x \subseteq (0T)_{0x}$.

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